Solution Sheet 8

Exercise 1. Let $\pi : (X, x_0) \to (Y, y_0)$ be a covering map between connected, path-connected, locally path-connected topological spaces. Set

$$G := \pi_1(Y, y_0), \qquad H := \pi_*\big(\pi_1(X, x_0)\big) \leq G,$$

and write $\mathrm{Deck}(\pi)$ for the group of deck (covering) transformations. Let $F = \pi^{-1}(y_0)$ be the fiber over y_0. Given a loop $\gamma : [0,1] \to Y$ based at y_0, and a point $x \in F$, there is a unique lift $\widetilde{\gamma}_x$ of γ starting at x, i.e. $\widetilde{\gamma}_x(0) = x$ and $\pi \circ \widetilde{\gamma}_x = \gamma$. The endpoint of the lift $\widetilde{\gamma}_x$ lands again in the fiber F. Therefore loops in Y induce permutations in the fiber F. Taking homotopy classes, this defines the monodromy representation

$$\rho : G \to \mathrm{Sym}(F), \quad \rho([\gamma])(x) := \widetilde{\gamma}_x(1) \in F.$$

(1) Show that $\rho(G)$ acts transitively on the fiber F.
(2) Show that $H = \mathrm{Stab}_G(x_0) = \{g \in G : \rho(g)(x_0) = x_0\}$.
(3) Show that the coset space G/H is in bijection with F.
(4) Show that

$$\ker(\rho) = \bigcap_{x \in F} \mathrm{Stab}_G(x) = \bigcap_{g \in G} gHg^{-1}.$$

(5) Show that if a deck transformation $\tau \in \mathrm{Deck}(\pi)$ fixes some $x \in X$, then $\tau = \mathrm{id}_X$. Show that every $\tau \in \mathrm{Deck}(\pi)$ induces a bijection $\tau|_F : F \to F$ and conclude that

$$\Phi : \mathrm{Deck}(\pi) \to \mathrm{Sym}(F), \qquad \Phi(\tau) = \tau|_F,$$

is injective.
(6) Show that $\Phi\big(\mathrm{Deck}(\pi)\big)$ is the centralizer of monodromy, that is,

$$\begin{aligned}\Phi\big(\mathrm{Deck}(\pi)\big) &= \mathrm{Cent}_{\mathrm{Sym}(F)}\big(\rho(G)\big) \\ &= \{\sigma \in \mathrm{Sym}(F) : \sigma\rho(g) = \rho(g)\sigma \text{ for all } g \in G\}.\end{aligned}$$

Solution 1.

(1) Let $x, y \in F = \pi^{-1}(y_0)$. Since X is path-connected, choose a path $\sigma : [0,1] \to X$ with $\sigma(0) = x$ and $\sigma(1) = y$. Let $\gamma = \pi \circ \sigma$. This is a loop at y_0. By the unique path lifting property, the lift of γ starting at x is σ, so

$$\rho([\gamma])(x) = \widetilde{\gamma}_x(1) = \sigma(1) = y.$$

This shows that $\rho(G)$ acts transitively on F.

(2) If $g \in H = \pi_*\big(\pi_1(X, x_0)\big)$, then there exists a loop $\tilde{\gamma}$ in X based at x_0 with $\pi_*([\tilde{\gamma}]) = [\pi \circ \tilde{\gamma}] = g$. The projection $\gamma = \pi \circ \tilde{\gamma}$ represents g. By uniqueness of path lifting, the lift of γ starting at x_0 is $\tilde{\gamma}$. Hence its endpoint is

$$\rho(g)(x_0) = \tilde{\gamma}_{x_0}(1) = x_0.$$

Conversely, if $\rho(g)(x_0) = x_0$, pick a loop γ at y_0 representing g. By assumption, the lift $\tilde{\gamma}_{x_0}$ of γ starting at x_0 ends at x_0. Hence $\tilde{\gamma}_{x_0}$ is a loop at x_0. The induced map on fundamental groups is then

$$\pi_*([\tilde{\gamma}_{x_0}]) = [\pi \circ \tilde{\gamma}_{x_0}] = [\pi] = g,$$

thus $g \in H = \pi_*\big(\pi_1(X, x_0)\big)$.

(3) We define $\psi : G/H \to F$ by $\psi(gH) = \rho(g)(x_0)$. This is well-defined, because if $g_1 H = g_2 H$, then $g_2^{-1} g_1 \in H = \mathrm{Stab}_G(x_0)$, thus

$$\rho(g_1)(x_0) = \rho(g_2)\rho(g_2^{-1} g_1)(x_0) = \rho(g_2)(x_0),$$

so $\psi(g_1H) = \psi(g_2H)$. The map is surjective because $\rho(G)$ acts transitively on F. It is also injective: if $\psi(g_1H) = \psi(g_2H)$, then $\rho(g_1)(x_0) = \rho(g_2)(s_0)$. Thus $\rho(g_2^{-1}g_1)(x_0) = x_0$, so $g_2^{-1}g_1 \in H = \mathrm{Stab}_G(x_0)$, that is $g_1H = g_2H$.

(4) An element $g \in \ker(\rho)$ fixes every $x \in F$, so $g \in \mathrm{Stab}_G(x)$ for all x. Since the action is transitive, for any $x \in F$ there exists $g \in G$ with $x = \rho(g)(x_0)$. Then it is easily shown that

$$\mathrm{Stab}_G(x) = g\,\mathrm{Stab}_G(x_0)\,g^{-1} = gHg^{-1}.$$

(5) Let $\tau \in \mathrm{Deck}(\pi)$ and suppose $\tau(\tilde{x}) = \tilde{x}$ for some $\tilde{x} \in X$. For any $y \in X$ choose a path $\widetilde{\alpha} : [0,1] \to X$ with $\widetilde{\alpha}(0) = \tilde{x}$ and $\widetilde{\alpha}(1) = y$. Set $\alpha := \pi \circ \widetilde{\alpha}$. Then $\widetilde{\alpha}$ is a lift of α starting at $\tilde{x}$. Since $\pi \circ \tau = \pi$ and $\tau(\tilde{x}) = \tilde{x}$, the path $\tau \circ \widetilde{\alpha}$ is also a lift of α starting at $\tilde{x}$. By uniqueness of lifts, this forces $\tau \circ \widetilde{\alpha} = \widetilde{\alpha}$ and hence $\tau(y) = y$. Because y was arbitrary, $\tau = \mathrm{id}_X$.

If $\tau \in \mathrm{Deck}(\pi)$ and $x \in F$, then

$$\pi(\tau(x)) = (\pi \circ \tau)(x) = \pi(x) = y_0,$$

so $\tau(x) \in F$. Since τ^{-1} is also a covering map, the same argument shows that $F \subset \tau(F)$. Since τ is a bijection $X \to X$ and $F = \tau(F)$, its restriction $\tau|_F : F \to F$ is a bijection, that is, $\Phi(\tau) \in \mathrm{Sym}(F)$.

To show that Φ is injective, suppose $\Phi(\tau) = \mathrm{id}_F$. Then τ fixes every point of F. By the fixed–point statement above this implies that $\tau = \mathrm{id}_X$.

(6) Let $\tau \in \mathrm{Deck}(\pi)$ and let γ be a loop based at y_0. For $x \in F$, $\widetilde{\gamma}_x$ is the lift of γ starting at x. Then $\tau \circ \widetilde{\gamma}_x$ is a lift of γ starting at $\tau(x)$, so by uniqueness we have

$$\tau \circ \widetilde{\gamma}_x = \widetilde{\gamma}_{\tau(x)}.$$

Hence for $x \in F$ we get

$$\Phi(\tau)\rho([\gamma])(x) = \tau(\widetilde{\gamma}_x(1)) = \widetilde{\gamma}_{\tau(x)}(1) = \rho([\gamma])\Phi(\tau)(x),$$

so $\Phi(\tau) \in \mathrm{Cent}_{\mathrm{Sym}(F)}\big(\rho(G)\big)$. For the converse, let $\sigma \in \mathrm{Cent}_{\mathrm{Sym}(F)}\big(\rho(G)\big)$. We need to construct a deck transformation $\tau : X \to X$ such that $\tau|_F = \sigma$. For $x \in X$, pick a path α in Y from y_0 to $\pi(x)$. We now define $\tau(x)$ to be the endpoint of the unique lift of α starting at $\sigma(x_0)$:

$$\tau(x) := \widetilde{\alpha}_{\sigma(x_0)}(1).$$

- *Well-defined.* This definition is independent of the choice of path α and starting point $x_0 \in F$: Suppose we have two pairs (x_0, α) and (x_0', α') with $\widetilde{\alpha}(1) = \widetilde{\alpha}'(1) = x$. The loop $\beta = \alpha'^{-1} \cdot \alpha$ based at y_0 lifts to a path in X from x_0 to x_0', that is, $\rho([\beta])(x_0) = x_0'$. Since σ centralizes $\rho(G)$, we have

  $$\rho([\beta])(\sigma(x_0)) = \sigma(\rho([\beta])(x_0)) = \sigma(x_0'). \tag{1}$$

  Now note that $\beta \cdot \alpha'$ is homotopic to α (relative endpoints). We lift both paths starting at $\sigma(x_0)$. By the homotopy lifting property, the lifts end at the same point:

  $$\widetilde{\alpha}_{\sigma(x_0)}(1) = \widetilde{\beta \cdot \alpha'}_{\sigma(x_0)}(1).$$

  But the lift of $\beta \cdot \alpha'$ is obtained by first lifting β (ending at $\rho([\beta])(\sigma(x_0))$), and then lifting α'. Hence by (1) we get

  $$\widetilde{\alpha}_{\sigma(x_0)}(1) = \widetilde{\beta \cdot \alpha'}_{\sigma(x_0)}(1) = \widetilde{\alpha}'_{\rho([\beta])(\sigma(x_0))}(1) = \widetilde{\alpha}'_{\sigma(x_0')}(1).$$

- $\pi \circ \tau = \pi$. For $x \in X$, take α from y_0 to $\pi(x)$. By construction we have

  $$\pi(\tau(x)) = \pi(\widetilde{\alpha}_{\sigma(x_0)}(1)) = (\pi \circ \widetilde{\alpha}_{\sigma(x_0)})(1) = \alpha(1) = \pi(x).$$

- $\tau|_F = \sigma$. If $x \in F = \pi^{-1}(y_0)$, take α to be the constant path at y_0, and $x_0 = x$. With these choices we have
$$\tau(x) = \widetilde{\alpha}_{\sigma(x)}(1) = \sigma(x).$$

Exercise 2. Let F be a set and let $A \leq \mathrm{Sym}(F)$ act transitively on F. Let $C = \mathrm{Cent}_{\mathrm{Sym}(F)}\big(A\big)$ be the centralizer of A in $\mathrm{Sym}(F)$. Show that

1. C acts freely on F.
2. C acts transitive on F if and only if A acts freely on F.

Solution 2.

1. Fix $x \in F$ and suppose $c \in C$ with $cx = x$. By transitivity, for any $y \in F$ we can choose $a \in A$ with $ax = y$. Thus
$$cy = cax = acx = ax = y.$$
Thus c fixes every point, hence c is the identity permutation. Thus the stabilizer of any point $x \in F$ is trivial, that is, the action is free.
2. Suppose C is transitive and A not free. Then there is $1 \neq a \in A$ and $x \in F$ with $ax = x$. For any y in the C-orbit of x we have
$$y := cx = cax = acx = ay,$$
thus a fixes the entire C-orbit of x. By transitivity of C, this orbit is all of F. Therefore a is the identity permutation, a contradiction.

   Suppose A is free and fix $x_0 \in F$. We need to show that the C-orbit of x_0 is all of F. Because A is free and transitive, we can write $y \in F$ uniquely as $y = bx_0$ for $b \in A$. For each $a \in A$ we define $r_a \in \mathrm{Sym}(F)$ by $r_a(y) := r_a(bx_0) := bax_0$. For any $g \in A$ and $y = bx_0$ we have
$$(r_a \circ g)(y) = r_a((gb)x_0) = (gba)x_0 = g(r_a(bx_0)) = (g \circ r_a)(y)$$
Thus $g \circ r_a = r_a \circ g$ for all $g \in A$, thus $r_a \in C$. Moreover, for any $y \in F$ there is a unique $a \in A$ with $y = ax_0$ and also
$$r_a(x_0) = r_a(ex_0) = ax_0 = y.$$
Thus every y lies in the C-orbit of x_0, hence C is transitive.

Exercise 3. Let $\pi : (X, x_0) \to (Y, y_0)$ be a covering map between connected, path-connected, locally path-connected topological spaces. Set
$$G := \pi_1(Y, y_0), \qquad H := \pi_*\big(\pi_1(X, x_0)\big) \leq G,$$
and write $\mathrm{Deck}(\pi)$ for the group of deck (covering) transformations. Let $d = |\pi^{-1}(y_0)|$ and let $\rho : G \to \mathrm{Sym}(\pi^{-1}(y_0)) \cong S_d$ be the monodromy representation on the fiber $F = \pi^{-1}(y_0)$. Prove that the following are equivalent (any may be taken as the definition of a Galois (regular/normal) covering):

1. $\mathrm{Deck}(\pi)$ acts transitively on the fiber F.
2. H is a normal subgroup of G.
3. There exists a free, proper action of a discrete group Γ on X with $Y \cong X/\Gamma$ and π a quotient map.
4. The monodromy image $\rho(G) \subset S_d$ acts freely and transitively on the fiber F.

When these equivalent conditions hold, we have group isomorphisms
$$\mathrm{Deck}(\pi) \cong \Gamma \cong G/H \cong \rho(G),$$
and $|\mathrm{Deck}(\pi)| = d$.

Solution 3.

(1) $\iff$ (4) From Exercise 1 and 2 we have

$$\begin{aligned}\operatorname{Deck}(\pi) \text{ is transitive on } F &\iff \operatorname{Cent}_{\operatorname{Sym}(F)}\big(\rho(G)\big) \text{ is transitive on } F\\ &\iff \rho(G) \text{ is free on } F.\end{aligned}$$

(1) $\iff$ (3) Assume (1). Take $\Gamma = \operatorname{Deck}(\pi)$ and endow Γ with the discrete topology. The action of Γ on X is:

- free: Deck transformations are fixed-point free.
- proper: Fix $x \in X$. Choose an evenly covered neighbourhood U of $\pi(x)$ and let V be a sheet with $x \in V$. For any $\tau \in \Gamma$, $\tau(V)$ is another sheet over U. Since sheets over U are disjoint, $\tau(V) \cap V \neq \emptyset$ implies that $\tau(V) = V$. The restriction $\pi|_V : V \to U$ is a homeomorphism, and when we apply its inverse to both sides of $\pi \circ \tau = \pi$ we get $\tau(z) = z$ for all $z \in V$. A deck transformation that fixes one point is the identity. Hence all nontrivial deck transformations move V away from itself, implying properness.

Note that deck transformations preserve fibers. Furthermore, transitivity on one fiber implies transitivity on every fiber. Thus the $\operatorname{Deck}(\pi)$-orbit of any $x \in X$ is the fiber $F_{\pi(x)}$. Let $q : X \to X/\operatorname{Deck}(\pi)$ be the quotient map. Because fibers of q equal fibers of π, there is a unique bijection $\bar{\pi} : X/\operatorname{Deck}(\pi) \to Y$ with $\bar{\pi} \circ q = \pi$. Moreover, $\bar{\pi}$ is a homeomorphism, so $X/\operatorname{Deck}(\pi) \cong Y$ and π is a quotient map with $\Gamma = \operatorname{Deck}(\pi)$. Conversely, for any $\gamma \in \Gamma$ we have $\pi(\gamma x) = \pi(x)$, so γ is a deck transformation. Fibers of π are exactly Γ-orbits, so Γ acts transitively on each fiber. Since $\Gamma \leq \operatorname{Deck}(\pi)$, the deck group acts transitively on each fiber as well.

(2) $\iff$ (4) Assume (4). Since $\rho(G)$ acts freely, $\operatorname{Stab}_{\rho(G)}(x_0) = \rho(H) = \{1\}$. Hence $H \subset \ker(\rho)$. By Exercise 1 we have $\ker(\rho) \subset H$, hence $H = \ker(\rho) = \bigcap_{g \in G} gHg^{-1}$. This implies that H is normal in G. Conversely, if H is normal, then $H = \ker(\rho) = \bigcap_{g \in G} gHg^{-1}$. Thus every $h \in H$ satisfies $\rho(h) = 1$, that is $\rho(H) = \{\mathrm{id}\}$. Hence $\operatorname{Stab}_{\rho(G)}(x_0) = \{\mathrm{id}\}$ and the action is free.

The only isomorphism we still need to justify is $\operatorname{Deck}(\pi) \cong \rho(G)$. Fix $x_0 \in F$ and define $\psi : \rho(G) \to \operatorname{Deck}(\pi)$ by $\psi(a) =$ the unique deck transformation τ with $\tau(x_0) = a(x_0)$. One can check that this is well-defined and bijective (injective by freeness and surjective by transitivity). It is an anti-homomorphism: Take $a, b \in \rho(G)$. Because deck transformations commute with monodromy on the fiber (by Exercise 1) we get

$$(\psi(a) \circ \psi(b))(x_0) = \psi(a)(b(x_0)) = b(\psi(a)(x_0)) = b(a(x_0)) = (ba)(x_0).$$

Because the deck action on F is free and transitive, a deck transformation is determined by where it sends x_0, thus $\psi(a) \circ \psi(b) = \psi(ba)$, so ψ is an anti-homomorphism. To turn it into a homomorphism, we just precompose with inversion

$$\Psi : \rho(G) \to \operatorname{Deck}(\pi), \Psi(a) = \psi(a^{-1}).$$

Exercise 4. Let $f : X \to Y$ be a nonconstant holomorphic map between connected Riemann surfaces. Let $B \subset Y$ be the set of branch values and put $U := Y \setminus B$ and $X^\circ := f^{-1}(U)$. We say that f is a branched Galois (regular/normal) covering if the unbranched covering

$$f|_{X^\circ} : \ X^\circ \to U$$

is Galois (regular/normal). Let τ be a deck transformation of the unbranched covering $f|_{X^\circ} : X^\circ \to U$.

(1) Show that there exists a unique biholomorphism $\overline{\tau} \in \operatorname{Aut}(X)$ such that

$$\overline{\tau}|_{X^\circ} = \tau \ \text{ and } \ f \circ \overline{\tau} = f \ \text{ on } X.$$

Deduce that the deck group on X° and the extended deck group on X are isomorphic.

(2) Show that for every $x \in X$,

$$e_{\overline{\tau}(x)}(f) = e_x(f),$$

i.e. the ramification multiplicity is invariant under deck transformations.

(3) Now assume that f is Galois. Show that for each $b \in B$, the (extended) deck group $\mathrm{Deck}(f)$ acts transitively on the branched fiber $f^{-1}(b)$. Deduce that for each $b \in B$, all points in $f^{-1}(b)$ have the same ramification multiplicity.

Solution 4. (1) Fix $b \in B$ and a small disk D around b containing no other branch values. Then

$$f^{-1}(D \setminus \{b\}) = \bigsqcup_{x \in f^{-1}(b)} A_x,$$

where each A_x is a punctured disk around x. Since $f \circ \tau = f$ on X°, τ permutes the components, say $\tau(A_x) = A_{x'}$. Because

$$f|_{A_x} = f|_{A_{x'}} \circ \tau|_{A_x},$$

and $\tau|_{A_x}$ is a biholomorphism, we have

$$e_x = \deg(f|_{A_x}) = \deg(f|_{A_{x'}}) = e_{x'} =: e.$$

We can choose charts $u : U_x \to \mathbb{D}$, $u' : U_{x'} \to \mathbb{D}$, and $v : V_b \to \mathbb{D}$ with $u(x) = u'(x') = v(b) = 0$ such that

$$v \circ f \circ u^{-1}(z) = z^e, \qquad v \circ f \circ (u')^{-1}(w) = w^e.$$

On $U_x^* = U_x \setminus \{x\}$ we have $f \circ \tau = f$. Define

$$\tilde{\tau} := u' \circ \tau \circ u^{-1} : \mathbb{D}^* \to \mathbb{D}^*.$$

Then, for $z \in \mathbb{D}^*$, we have

$$(v \circ f \circ (u')^{-1})\big(\tilde{\tau}(z)\big) = v \circ f \circ \tau \circ u^{-1}(z) = (v \circ f \circ u^{-1})(z),$$

hence

$$(\tilde{\tau}(z))^e = z^e.$$

Set $\beta(z) := \tilde{\tau}(z)/z$. This is holomorphic and nowhere zero on $\mathbb{D}^*$ and satisfies $\beta(z)^e = 1$, so $\beta = \zeta$ for some ζ with $\zeta^e = 1$. Therefore

$$\tilde{\tau}(z) = u' \circ \tau \circ u^{-1}(z) = \zeta z \qquad (z \in \mathbb{D}^*),$$

that is, in these coordinates τ is a rotation by an e-th root of unity. Hence

$$|\tilde{\tau}(z)| = |z| \to 0 \quad \text{as } z \to 0.$$

Because $\tilde{\tau}$ is holomorphic on $\mathbb{D}^*$ and bounded near 0, Riemann's removable singularity theorem implies that $\tilde{\tau}$ extends uniquely to a holomorphic map on $\mathbb{D}$ by setting $\tilde{\tau}(0) = 0$. Then we define on U_x the holomorphic extension map

$$\overline{\tau}_x := (u')^{-1} \circ (\zeta\,\cdot) \circ u : U_x \to X,$$

which agrees with τ on U_x^* and satisfies $f \circ \overline{\tau}_x = f$ on U_x. Indeed:

- Take $p \in U_x^*$ and set $z = u(p)$. Then $u'(\tau(p)) = \tilde{\tau}(z) = \zeta z$, so

$$\tau(p) = (u')^{-1}(\zeta z) = (u')^{-1} \circ (\zeta\,\cdot) \circ u(p) = \overline{\tau}_x(p).$$

- For $p \in U_x$ and $z = u(p)$ we have

$$\begin{aligned} v \circ f \circ \bar{\tau}_x(p) &= v \circ f \circ (u')^{-1}(\zeta z) \\ &= (\zeta z)^e \\ &= z^e \\ &= \big(v \circ f \circ u^{-1}\big)(z) \\ &= v \circ f(p). \end{aligned}$$

  Because v is a biholomorphism this implies that $f \circ \bar{\tau}_x = f$ on U_x.

Doing this at every branch preimage yields a holomorphic map $\overline{\tau} : X \to X$ with $\overline{\tau}|_{X^\circ} = \tau$ and $f \circ \overline{\tau} = f$. Applying the same construction to τ^{-1} yields a holomorphic inverse of $\bar{\tau}$. The homomorphism

$$\operatorname{Deck}(f) \to \operatorname{Deck}(f|_{X^\circ}), \quad \overline{\tau} \mapsto \overline{\tau}|_{X^\circ}.$$

is injective (by the identity principle) and surjective (by the above extension). Therefore the deck group on X° and the extended deck group on X are canonically identified.

(2) We use the same notation as above. There are charts $u : U_x \to \mathbb{D}$, $u' : U_{x'} \to \mathbb{D}$, $v : V_{f(x)} \to \mathbb{D}$ with

$$v \circ f \circ u^{-1}(z) = z^{e_x}, \qquad v \circ f \circ (u')^{-1}(w) = w^{e_{x'}}.$$

We can define a holomorphic map $h := u' \circ \overline{\tau} \circ u^{-1}$ with $h(0) = 0$ and $h(z) = az + O(z^2)$ with $a \neq 0$. From $f \circ \overline{\tau} = f$ we get

$$z^{e_x} = (v \circ f \circ u^{-1})(z) = (v \circ f \circ (u')^{-1})\big(h(z)\big) = (h(z))^{e_{x'}}.$$

Comparing lowest orders in z gives $e_x = e_{x'}$ (and forces $a^{e_x} = 1$).

(3) Fix $b \in B$ and a disk D around b containing no other branch values and let $D^* = D \setminus \{b\}$. Then

$$f^{-1}(D^*) = \bigsqcup_{x \in f^{-1}(b)} A_x,$$

with each A_x a punctured disk around x and $f|_{A_x} : A_x \to D^*$ an unbranched covering. For $x_1, x_2 \in f^{-1}(b)$ choose $y \in D^*$ and points $p \in A_{x_1} \cap f^{-1}(y)$, $q \in A_{x_2} \cap f^{-1}(y)$. Since $f|_{X^\circ}$ is Galois, $G = \operatorname{Deck}(f|_{X^\circ})$ acts transitively on each fiber over D^*. Thus there exists $\tau \in G$ with $\tau(p) = q$. By part (1), τ extends uniquely to $\overline{\tau} \in \operatorname{Aut}(X)$ with $f \circ \overline{\tau} = f$. This forces $\overline{\tau}(x_1) = x_2$ and thus $\operatorname{Deck}(f)$ is transitive on $f^{-1}(b)$. By part (2) we have

$$e_{x_1}(f) = e_{\overline{\tau}(x_1)}(f) = e_{x_2}(f),$$

so all points of $f^{-1}(b)$ have the same ramification multiplicity.

Exercise 5. (for credit, due on 16 November) *(5 points)*
Fix $n \in \mathbb{Z}_{\geq 1}$ and consider the map $f_n : \mathbb{P}^1 \to \mathbb{P}^1$ defined by

$$f_n(z) = \frac{z^{2n} + 1}{z^n}.$$

Show that f_n is a branched Galois covering whose group of deck transformations is isomorphic to the dihedral group:

$$\operatorname{Deck}(f_n) \cong D_{2n} = \langle \sigma, \rho : \rho^n = \sigma^2 = 1, \sigma\rho\sigma = \rho^{-1} \rangle.$$

Solution 5. We determine the ramifications points of f_n as in previous exercise sheets. They are

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- 0 and ∞ with ramification multiplicity n, both map to ∞;
- the n-th roots of 1 with ramification multiplicity 2, mapping to 2;
- the n-th roots of -1 with ramification multiplicity 2, mapping to -2.

From the previous exercise we know that any deck transformation of the unbranched cover $f_n|_{X^\circ}$ extends uniquely to $\bar{\tau} \in \mathrm{Aut}(\mathbb{P}^1) = PGL_2(\mathbb{C})$ and $f_n \circ \bar{\tau} = f_n$. We also know that any deck transformation must preserve fibers and ramification multiplicities; we will use these constraints to determine the deck group. A Möbius map preserving the fiber $f_n^{-1}(\infty) = \{0, \infty\}$ has the form

$$\phi(z) = \lambda z \quad \text{or} \quad \phi(z) = \frac{\lambda}{z}$$

for $\lambda \in \mathbb{C}^\times$. For ϕ to be a deck transformation of $f_n(z) = z^n + z^{-n}$ we further impose the constraint $f_n \circ \phi = f_n$. We compute:

- If $\phi(z) = \lambda z$, then
$$(f_n \circ \phi)(z) = f_n(\lambda z) = \lambda^n z^n + \lambda^{-n} z^{-n}.$$
This equals $z^n + z^{-n}$ for all z precisely when $\lambda^n = 1$.
- If $\phi(z) = \lambda / z$, then
$$(f_n \circ \phi)(z) = f_n(\lambda/z) = \lambda^n z^{-n} + \lambda^{-n} z^n,$$
which again equals $f_n(z)$ if and only if $\lambda^n = 1$.

Therefore the Möbius transformations preserving the fiber $f_n^{-1}(\infty) = \{0, \infty\}$ are precisely the $2n$ maps

$$z \mapsto \zeta z, \quad z \mapsto \frac{\zeta}{z}$$

with $\zeta^n = 1$. Note that both maps $z \mapsto \zeta z, z \mapsto \frac{\zeta}{z}$ permute the sets $f_n^{-1}(2) = \{z^n = 1\}$ and $f_n^{-1}(-2) = \{z^n = -1\}$, so no new restrictions arise. Now pick a primitive n-th root ω and set

$$\rho(z) = \omega z, \quad \sigma(z) = \frac{1}{z}.$$

The $2n$ maps from before are precisely

$$\rho^k(z) = \omega^k z \quad \text{and} \quad (\sigma\rho^k)(z) = \frac{1}{\omega^k z} = \frac{\omega^{-k}}{z}$$

for $k = 0, \ldots, n-1$. These maps form a group with relations

$$\rho^n = \sigma^2 = 1, \quad \sigma\rho\sigma = \rho^{-1},$$

hence $\langle \sigma, \rho \rangle \cong D_{2n}$. A generic fiber consists of $2n$ elements and on X° the action of $\langle \sigma, \rho \rangle$ is free. Indeed:

- If $\rho^k(z) = z$, then $\omega^k z = z$. For $z \neq \{0, \infty\}$ this forces $\omega^k = 1$, that is, $k \equiv 0 \pmod{n}$. Thus no nontrivial ρ^k fixes a point of X°.
- If $(\sigma\rho^k)(z) = z$, then $z^2 = \omega^{-k}$ and so $z^{2n} = 1$. These points are removed in X°, so no $\sigma\rho^k$ fixes a point of X°.

Therefore a group of size $2n$ acts freely on a fiber of size $2n$, which implies that the action is transitive. Hence f_n is a Galois cover with

$$\mathrm{Deck}(f_n) \cong D_{2n}, \quad \deg(f_n) = 2n = |\mathrm{Deck}(f_n)|.$$

Exercise 6. Find a branched covering $f : \mathbb{P}^1 \to \mathbb{P}^1$ that is not Galois.

Solution 6. For example $f(z) = z + z^k$ is not a Galois covering for any $k \geq 3$. The group of deck transformations is trivial, but the degree of f is $k \geq 3$.